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CROSSING FAMILIES

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Given a set of points in the plane, a crossing family is a collection of line segments, each joining two of the points, such that any two line segments intersect internally. Two sets A and B of points in the plane are mutually avoiding if no line subtended by a pair of points in A intersects the convex hull of B, and vice versa. We show that any set of n points in general position contains a pair of mutually avoiding subsets each of size at least $\sqrt{n/12}$. As a consequence we show that such a set possesses a crossing family of size at least $\sqrt{n/12}$, and describe a fast algorithm for finding such a family.

1. Introduction

Consider n points in the plane in general position (no three points collinear). We say that a collection of line segments, each joining two of the given points, is a *crossing family* if every two segments intersect internally. In a natural variation the points belong to two color classes, and each segment of the crossing family joins points of different colors. We say that two equal-sized disjoint sets A and B can be *crossed* if there exists a crossing family exhausting A and B in which each line segment connects a point in A with a point in B.

In this paper we study crossing families. In Section 2 we show that $\Omega(\sqrt{n})$-size crossing families exist in both the colored and uncolored versions of the problem. Our proof is constructive, and yields an algorithm which can be implemented to find such a family in time $O(n \log n)$.

We obtain the result on crossing families by finding sets of points which are *mutually avoiding*. Say that a set A *avoids* a set B if no line (not line segment) subtended by a pair of points in A intersects the convex hull of B. This means that every vertex in B "sees" the points of A in the same order. The sets A and B are *mutually avoiding* if A avoids B and B avoids. We show how to find mutually avoiding sets of size $\Omega(\sqrt{n})$. Valtr [7] has shown that this is best possible up to the

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constant. The result on crossing families then follows from showing that if a pair of sets A, B are mutually avoiding and of equal cardinalities then they can be crossed.

In Section 3 we characterize which pairs of sets are mutually avoiding and which can be crossed. This characterization shows that mutual avoidance is a much stronger notion that crossability, and supports our belief that the true size of a maximum crossing family grows more quickly than $\sqrt{n}$. (It could even be linear.) In Section 4 we show that the crossing family problem is equivalent to the problem of finding a collection of line segments which are pairwise "parallel": i.e. the lines subtended by any pair of segments intersect beyond the segments.

The notions of avoidance and mutual avoidance extend naturally to higher dimensions: if A and B are sets of points in $\mathbb{R}^d$, then A avoids B if no hyperplane subtended by d points in A intersects the convex hull of B. In Section 5 we show that polynomial-sized mutually avoiding sets exist in arbitrary dimensions.

Several researchers have considered problems involving configurations of m line segments among n points in the plane. Alon and the second author [1] showed that if $m \geq 6n-5$ then there are always three mutually disjoint line segments. This was extended by the sixth author and Töröcsik [6] who showed that if $m > k^4 n$ then there are $k+1$ mutually disjoint line segments. Capoyleas and the sixth author [2] showed that for $k \leq n/2$ if the points are in convex position and $m > (k-1)(2n+1-2k)$, then there is a crossing family of size k, and that this is best possible.

2. Construction of an $\Omega(\sqrt{n})$ crossing family

In this section we show, given n points in general position in the plane, how to find a pair of mutually avoiding sets X' and Y' of size $\Omega(\sqrt{n})$. This is achieved by finding subsets X and Y such that X avoids Y, and then subsets $X' \subseteq X$ and $Y' \subseteq Y$ such that Y' avoids X'. Since a pair of equal-sized mutually avoiding sets can be crossed (see Corollary 1), we thus obtain a crossing family of cardinality $\min(|X'|, |Y'|)$.

We use the following well-known results:

Lemma 1. *For any line $\mathcal{L}$ in the plane and finite set of points, it is possible to find another line $\mathcal{M}$ which simultaneously splits the points in both halfplanes in any desired proportions.*

Lemma 2. [4] *Among any sequence of real numbers of length n, there is either an ascending or a descending subsequence of length $\sqrt{n}$.*

We will work in the two-color case, where X is to be chosen from among $n/2$ blue, and Y from among $n/2$ red points.

Theorem 1.
(i) *Given $n/2$ red and $n/2$ blue points, there exists a crossing family of size at least $\sqrt{n/24}$.*
(ii) *Given n uncolored points, there exists a crossing family of size at least $\sqrt{n/12}$.*

Proof. Our strategy in proving (i) has three steps.

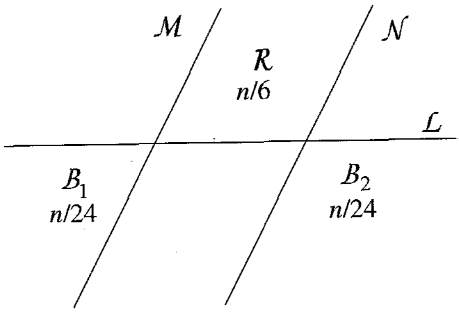

Fig. 1. The H-picture.

Step 1. This is a preliminary step where the plane is partitioned by three lines (as depicted in Figure 1) so that certain regions have linearly many points of particular colors.

Specifically, first find a line $\mathscr{L}$ such that at least $n/4$ of the reds are on one side and at least $n/4$ blues on the other by moving a horizontal $\mathscr{L}$ down from $y=+\infty$ until $n/4$ of the first color, say red, are above it. Discard the blue points above $\mathscr{L}$ and the red points below it. Second, use Lemma 1 to find a line $\mathscr{M}$ such that exactly $n/24$ of the red and $n/24$ of the blue points are to the left of $\mathscr{M}$. Finally, take a line $\mathscr{N}$ parallel to $\mathscr{M}$ at $x=+\infty$ and move it to the left until $n/24$ of the first color, say blue, are on its right. See Figure 1. The region $\mathscr{R}$ contains at least $n/6$ red points, and the regions $\mathscr{B}_1$ and $\mathscr{B}_2$ both at least $n/24$ blue points.

Step 2. For convenience, apply an affine transformation such that $\mathscr{M}$ and $\mathscr{N}$ are vertical. Order the reds in $\mathscr{R}$ from left to right. By Lemma 2 there exists either an ascending or a descending subsequence R of length $\sqrt{n/6}$. Without loss of generality assume that R is *descending*. Then observe that R avoids $\mathscr{B}_1$.

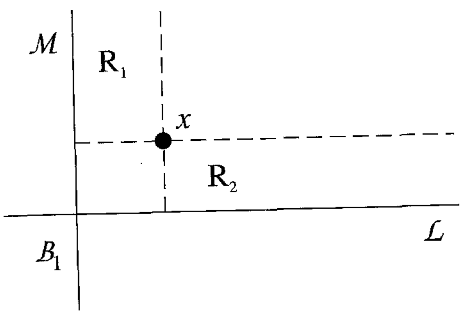

Fig. 2. x splits R into two parts.

Step 3. Consider the middle points x of R breaking it into two parts R_1 and R_2, each a descending sequence of length $\sqrt{n/24}$. See Figure 2.

Consider the positions of the blue points in $\mathscr{B}_1$ expressed in polar coordinates (r,θ) with x as the origin (and θ measured counterclockwise), and order them as $\{b_i\}$ (for $i=1,\ldots,n/24$) in decreasing distance r_i from x. By Lemma 2 there exists a subsequence $B=\{b_{k_i}\}$ (for $i=1,\ldots,\sqrt{n/24}$) whose angles θ_{k_i} are either decreasing or increasing. Say they are *increasing*. We claim that B avoids R_1. For consider two points b_{k_i} and b_{k_j} of B with $i<j$. From the conditions on B it follows that b_{k_j} is to the right of b_{k_i} and below the line subtended by x and b_{k_1}. Thus the line spanned by b_{k_i} and b_{k_j} avoids the region containing R_1.

Applying Corollary 1, the theorem follows. The only change for the uncolored case (ii) is that $\mathscr{L}$ may be found without discarding half the points. ∎

The above procedure provides an $O(n\log n)$-time algorithm for constructing a crossing family since one can apply Lemmas 1 and 2 in this time. (For Lemma 1 cf. [3].)

3. A characterization

In this section we examine conditions which characterize when two sets can be crossed and when they are mutually avoiding.

Consider red points X and blue points Y separated by a line $\mathscr{L}$. We say a red point x sees a blue point y at rank i if y is the i^{th} blue point counterclockwise as seen from x. And vice versa. Then we say X and Y obey the *rank condition* if there exist labelings $x_1,\ldots,x_s$ and $y_1,\ldots,y_s$ of X and Y such that for all i, x_i sees y_i at rank i and vice versa. For the *strong rank condition*, the labelings must be such that x_i sees y_j at rank j for all i and j.

Proposition 1. *Let X and Y be s red and s blue points separated by a line. Then:*
(1) X and Y can be crossed if and only if they obey the rank condition.
(2) X and Y are mutually avoiding if and only if they obey the strong rank condition.

Since the strong rank condition implies the rank condition, this gives:

Corollary 1. *A pair of sets can be crossed if they are mutually avoiding and of equal cardinality.*

Proof. **(1)** Say the line $\mathscr{L}$ is vertical, with reds X on the left and blues Y on the right.

Assume first that X and Y can be crossed. Let $l_1,\ldots,l_s$ be the line segments of a complete crossing family in order of increasing slope. Label the red endpoint of l_i, x_i, and the blue endpoint y_i. Since $l_1,\ldots,l_{i-1}$ are of lesser slope that l_i, and intersect it, x_i sees $y_1,\ldots,y_{i-1}$ before it sees y_i. Similarly, x_i sees $y_{i+1},\ldots,y_s$ after y_i and thus it sees y_i at rank i. For the same reason, y_i sees x_i at rank i.

Assume now that there exist labeling x_i, y_i satisfying the rank condition. We prove by induction on s that the family $\{x_iy_i\}_i$ of line segments is a crossing family. The case $s=1$ is trivial.

Consider the line ℓ_s extending the segment x_sy_s. By the rank condition, $X-x_s$ and $Y-y_s$ lie on opposite sides of this line. Hence x_sy_s intersects x_iy_i if it intersects ℓ_i. Also, the slope of x_iy_i is less than that of x_sy_s for all $i<s$.

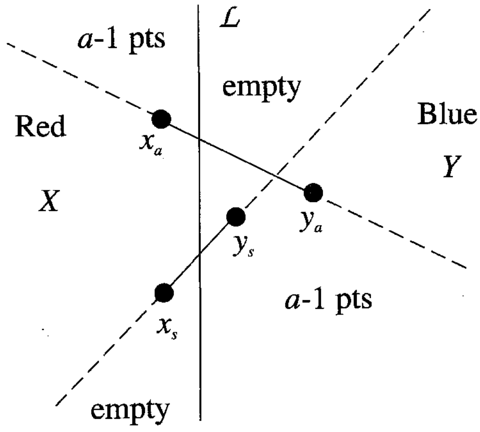

Fig. 3. The picture when $x_a y_a$ is above $x_s y_s$.

Let A be the set of all line segments that do not intersect $x_s y_s$; order the members of A with respect to their $\mathscr{L}$-intercepts. If A is nonempty then without loss of generality it contains a line segment whose $\mathscr{L}$-intercept is above that of $x_s y_s$; then choose a such that $x_a y_a$ has the highest $\mathscr{L}$-intercept in A. See Figure 3. Then the line ℓ_a extending $x_a y_a$ does not intersect $x_s y_s$. So there exist $a-1$ red points above ℓ_a and $a-1$ blue points below it. But y_s is among these blue points while x_s is not among the red ones. Thus there exists b so that x_b and y_b are both above ℓ_a. Then $x_b y_b$ does not intersect $x_s y_s$ and has a higher $\mathscr{L}$-intercept than $x_a y_a$, contrary to assumption. Hence A is empty, and $x_s y_s$ intersects all $x_i y_i$.

To prove that $x_i y_i$ intersects $x_j y_j$ for all $i < j < s$, we observe that since the slope of $x_s y_s$ is greater than that of any other segment, the rank condition is preserved upon deletion of x_s and y_s, with the same labeling $x_1, \ldots, x_{s-1}$ and $y_1, \ldots, y_{s-1}$. Hence by induction $\{x_i y_i\}_1^{s-1}$ is a crossing family.
(2) The order in which a point x sees two points y_1 and y_2 is determined by which side of the line through them x lies. If Y avoids X then each pair y_i, y_j is seen in the same order by all $x \in X$, and a similar conclusion holds if X avoids Y; hence if X and Y are mutually avoiding then they satisfy the strong rank condition.

If on the other hand the sets are not mutually avoiding, say Y does not avoid X, then there is a pair of points in Y subtending a line through the convex hull of X, and this pair is seen in a different order by the points of X to each side of that line. Hence the strong rank condition fails. ∎

4. Parallel families

We use the term "parallel" to describe a pair of nonintersecting line segments whose extensions intersect outside both segments. We say that a family of line segments is parallel if every pair of the segments is parallel.

The following can be shown in the manner of Proposition 1:

Proposition 2. *Let X and Y be two sets, of s points each, separated by a line. Then X and Y can be paired up to form a parallel family if and only if there exist labeling $x_1,\dots,x_s$ and $y_1,\dots,y_s$ of X and Y such that for all i, x_i sees y_i at rank i and y_i sees x_i at rank $s+1-i$. In particular, if X and Y are mutually avoiding then X and Y can be so paired.*

The problems of finding large parallel and large crossing families are equivalent:

Theorem 2. *Let $c(n)$ (resp. $p(n)$) denote the minimum number of segments in a maximum crossing (parallel) family, among all configurations of n blue and n red points separated by a line, with all $2n$ points in general position. Then $c(n)=p(n)$.*

Proof. Consider a configuration of points with n points either side of the y-axis and the transformation f given by $(x,y)\mapsto(1/x,y/x)$. This carries the points to a new configuration such that if segments b_1r_1 and b_2r_2 intersect and meet the y-axis, then segments $f(b_1)f(r_1)$ and $f(b_2)f(r_2)$ are parallel and meet the y-axis, and vice versa. ∎

5. Mutually avoiding sets in higher dimensions

In this section we show that there are polynomial-sized mutually avoiding sets in arbitrary dimensions. A hyperplane *stabs* a set in $\mathbb{R}^d$ if it intersects the convex hull of that set. The *stabbing number* of a collection of sets is the maximum number of sets that any hyperplane stabs. We use the following result of Matoušek:

Lemma 3. [5] *Let P be a set of n points in $\mathbb{R}^d$ and let $r\le n$. Then there exists a subset $P'\subseteq P$ of at least $n/2$ points and a partition $\{P_1,\dots,P_m\}$ of P' with $|P_i|=\lfloor n/r\rfloor$ for all i and with stabbing number $O(r^{1-1/d})$.*

Theorem 3. *Any set of n points in $\mathbb{R}^d$ contains a pair of mutually avoiding subsets each of size $\Omega(n^{1/(d^2-d+1)})$.*

Proof. Say we apply the above result with parameter r yielding a partition of P' into blocks $P_1,\dots,P_m$. Note that m is $\Theta(r)$. The points of each P_i generate $O(n^d/r^d)$ hyperplanes, and each of these hyperplanes stabs $O(r^{1-1/d})$ subsets. Thus there are at most $O(n^d r^{2-d-1/d})$ stabbings in all.

Associate every stabbing with the (unordered) pair of blocks consisting of the block generating the hyperplane, and the block which is stabbed by it. Since there are $\Theta(r^2)$ pairs of blocks, some pair has only $n^d r^{-d-1/d}$ such mutual stabbings. If r is chosen to be approximately $n^{(d^2-d)/(d^2-d+1)}$ then this pair has at most $n/2r$ mutual stabbings. Each stabbing is created by a hyperplane which can be eliminated by removing one point from one of the blocks. The depleted blocks are each of size at least $n/2r=\Omega(n^{1/(d^2-d+1)})$, and are mutually avoiding. ∎

We can use the above result to find an analogue of a polynomial-sized crossing family in $\mathbb{R}^d$: a collection of d-simplices such that every two simplices intersect and have disjoint vertex sets. We omit the construction.

6. Discussion

We believe our lower bound on the size of maximum crossing families can be improved. Our best upper bound is linear: at most $n/2$ points used for the uncolored case (for example for non-convex points) and at most $3n/8$ in the colored case. For the latter consider the arrangement of sixteen points in Figure 4. These ratios can be obtained for arbitrarily large n by splitting points suitably. Regarding the behavior of "generic" sets of points, we note that n points (colored or not) chosen at random in the unit disk, almost surely have a linear-sized crossing family. We omit the details.

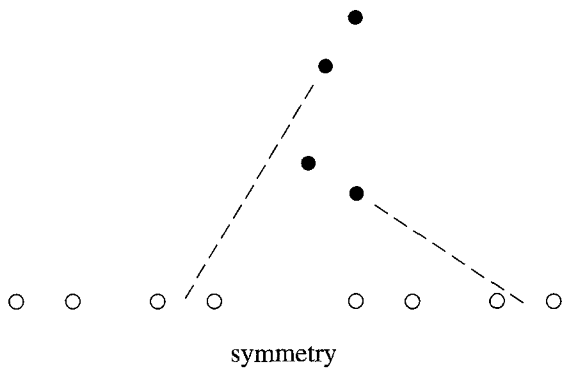

Fig. 4. Arrangement for upper bound for two colored case: 4 more black points are located symmetrically below the whites.

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